Filomat 29:3 (2015), 611–618
DOI 10.2298/FIL1503611M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

CR-Submanifolds with the Symmetric $\nabla\sigma$ in a Locally Conformal Kaehler Space Form

Koji Matsumoto[a]

[a] *2-3-65 Nishi-Odori, Yonezawa, Yamagata, 992-0059, Japan*

Abstract. In this paper, we consider CR-submanifolds with the symmetric $\nabla\sigma$ which is a generalization of parallel second fundamental form, in a locally conformal Kaehler space form. About the symmetric tensor field P defined in (1.7), we show that, in an anti-holomorphic submanifold in an l.c.K.-space form, P is diagonal with respect to an adapted frame and has two eigenfunctions (See Theorem 3.1). Finally, we consider the relation of the eigenfunctions of P and the Lee form (See Theorems 3.2 and 3.3).

1. Locally conformal Kaehler manifolds.

A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is called a locally conformal Kaehler (an l.c.K.-) manifold if each point $x \in \tilde{M}$ has an open neighbourhood U with a positive differentiable function $\rho : U \to \mathcal{R}$ such that $\tilde{g}^* = e^{-2\rho}\tilde{g}_{|U}$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is the almost complex structure, $\tilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\tilde{g}^*$, $\tilde{g}_{|U}$ is the restriction of $\tilde{g}$ to U and $\mathcal{R}$ is a real number space ([8] -[10],[13], etc.).

Remark 1.1. *We know that a typical example of a compact l.c.K.-manifold is a Hopf manifold which has no Kaehler structure ([11],[12]) and examples of non-compact case are in [7].*

Then the following useful proposition is wellknown ([8]);

Proposition 1.1. *A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is l.c.K.- if and only if there exists a global 1-form α which is called the Lee form satisfying*

$$J^2 = -I, \tag{1.1}$$

$$\tilde{g}(JV, JU) = \tilde{g}(V, U), \tag{1.2}$$

$$N_J(V, U) = 0, \tag{1.3}$$

$$d\alpha = 0 \quad (\alpha : closed), \tag{1.4}$$

$$(\tilde{\nabla}_V J)U = -\tilde{g}(\alpha^\sharp, U)JV + \tilde{g}(V, U)\beta^\sharp + \tilde{g}(JV, U)\alpha^\sharp - \tilde{g}(\beta^\sharp, U)V \tag{1.5}$$

2010 *Mathematics Subject Classification*. Primary 53C05; Secondary 53C40, 53A40
Keywords. Locally conformal Kaehler space form, anti-holomorphic submanifold, CR-submanifold, symmetric $\nabla\sigma$
Received: 13 November 2014; Accepted: 15 December 2014
Communicated by Ljubica Velimirović and Mića Stanković
Email address: tokiko_matsumoto@yahoo.com (Koji Matsumoto)

for any $V, U \in T\tilde{M}$, where $\tilde{\nabla}$ denotes the covariant differentiation with respect to $\tilde{g}$, $\alpha^\sharp$ is the dual vector field of α which is called the Lee vector field, the 1-form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β, $T\tilde{M}$ means the tangent bundle of $\tilde{M}$ and N_J denotes the Nijenhuis tensor with respect to J which is defined by

$$N_J(V, U) = [JV, JU] - J[JV, U] - J[V, JU] + J^2[V, U]([14]).$$

We write such a manifold $\tilde{M}(J, \tilde{g}, \alpha)$.

An l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature, that is, $\tilde{R}(JU, U, U, JU) = constant$ for any unit $U \in T\tilde{M}$, where $\tilde{R}$ is the Riemannian curvature tensor with respect to $\tilde{g}$. Then we know that the tensor $\tilde{R}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by ([8])

$$\begin{aligned}4\tilde{R}(W, Z, V, U) = c\{&\tilde{g}(W, U)\tilde{g}(Z, V) - \tilde{g}(W, V)\tilde{g}(Z, U) + \tilde{g}(JW, U)\tilde{g}(JZ, V) - \tilde{g}(JW, V)\tilde{g}(JZ, U)\\ &- 2\tilde{g}(JW, Z)\tilde{g}(JV, U)\} + 3\{P(W, U)\tilde{g}(Z, V) - P(W, V)\tilde{g}(Z, U) + \tilde{g}(W, U)P(Z, V)\\ &- \tilde{g}(W, V)P(Z, U)\} - \tilde{P}(W, U)\tilde{g}(JZ, V) + \tilde{P}(W, V)\tilde{g}(JZ, U) - \tilde{g}(JW, U)\tilde{P}(Z, V)\end{aligned} \tag{1.6}$$

for any $W, Z, V, U \in T\tilde{M}$, where P and $\tilde{P}$ are respectively defined by

$$P(V, U) = -(\tilde{\nabla}_V\alpha)U - \alpha(V)\alpha(U) + \frac{1}{2}\|\alpha\|^2\tilde{g}(V, U), \tag{1.7}$$

and

$$\tilde{P}(V, U) = P(JV, U) \tag{1.8}$$

for any $V, U \in T\tilde{M}$, where $\|\alpha\|$ is the length of the Lee vector field $\alpha^\sharp$ with respect to $\tilde{g}$, that is, $\|\alpha\|^2 = \tilde{g}(\alpha^\sharp, \alpha^\sharp)$.

Remark 1.2. *To get (1.6), we have to assume that the symmetric (0,2)-tensor P is hybrid or equivalently $\tilde{P}$ is skew-symmetric. This means that the Ricci tensor $\tilde{R}_1$ with respect to $\tilde{g}$ is hybrid.*

Remark 1.3. *We know that a Hopf manifold is an l.c.K.-space form with the parallel Lee form ($\nabla\alpha = 0$). And it has no hybrid P. But, we don't know the representation of the Riemannian curvature tensor of an l.c.K.-space form with non hybrid P.*

We write $\tilde{M}(c)$ an l.c.K.-space form with the constant holomorphic sectional curvature c.

2. *CR*-submanifolds in an l.c.K.-manifold.

In generally, between a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its Riemannian submanifold M, the Gauss and the Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{2.2}$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$, where σ is the second fundamental form, A_ξ is the shape operator with respect to ξ, $\nabla^\perp$ is the normal connection and $T^\perp M$ is the normal bundle of M([6]). The second fundamental form σ and the shape operator A are related by

$$\tilde{g}(A_\xi Y, X) = \tilde{g}(\sigma(Y, X), \xi)$$

for any $Y, X \in TM$ and $\xi \in T^\perp M$.

The Codazzi equation is given by

$$\{\tilde{R}(X,Y)Z\}^{\perp} = (\nabla_X\sigma)(Y,Z) - (\nabla_Y\sigma)(X,Z), \tag{2.3}$$

for any $X, Y, Z \in TM$, where $\{\tilde{R}(X,Y)Z\}^{\perp}$ denotes the normal part of $\tilde{R}(X,Y)Z$ and $(\nabla_X\sigma)(Y,Z)$ is defined by

$$(\nabla_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_XY,Z) - \sigma(Y,\nabla_XZ) \tag{2.4}$$

for any $X, Y, Z \in TM$ ([6]).

The tensor field $\nabla\sigma$ is said to be symmetric if $(\nabla_Z\sigma)(Y,X)$ is symmetric with respect to any $Z, Y, X \in TM$ and the second fundamental form σ is said to be parallel if it satisfies $\nabla\sigma = 0$.

Remark 2.1. *The above definitions mean that the normal part of $\tilde{R}(Z,Y)X$ is identically zero for any $Z, Y, X \in TM$, that is, the Codazzi equation is zero.*

Remark 2.2. *In a Riemannian manifold $\tilde{M}$, a symmetric (0,2) tensor T is said to be a Codazzi type if $(\nabla_XT)(Y,Z)$ is symmetric with respect to any $X, Y, Z \in T\tilde{M}$.*

Definition 2.1. *A submanifold M in an l.c.K.-manifold $\tilde{M}$ is called a CR-submanifold if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ on M satisfying the following conditions;*

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $\mathcal{D}^{\perp} : x \to \mathcal{D}^{\perp}_x \subset T_xM$ is totally real, i.e., $J\mathcal{D}^{\perp}_x \subset T^{\perp}_xM$ for each $x \in M$, where T_xM (resp. $T^{\perp}_xM$) denotes the tangent (resp. normal) vector space at x of M ([1]-[5],etc.).

In a CR-submanifold, the distribution $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) is called a *holomorphic* (resp. *totally real*) distribution.

If $\dim \mathcal{D}^{\perp}_x = 0$ (resp. $\dim \mathcal{D}_x = 0$) for each $x \in M$, then the CR-submanifold is a *holomorphic* (resp. *totally real*) submanifold. A CR-submanifold M is said to be *anti-holomorphic* if $J\mathcal{D}^{\perp}_x = T^{\perp}_xM$ for any $x \in M$.

For a CR-submanifold M of an almost Hermitian manifold $\tilde{M}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^{\perp}$ in the normal bundle $T^{\perp}M$. Then we have the following direct sum decomposition

$$T^{\perp}M = J\mathcal{D}^{\perp} \oplus \nu, \qquad J\mathcal{D}^{\perp} \perp \nu. \tag{2.5}$$

Remark 2.3. *By the definition of the distribution ν, a CR-submanifold in an l.c.K.-manifold is anti-holomorphic if $\nu_x = \{0\}$ for any $x \in M$.*

In a CR-submanifold M of an l.c.K.-manifold $\tilde{M}$, let be $\dim \mathcal{D} = 2p$, $\dim \mathcal{D}^{\perp} = q$, $\dim M = n$, $\dim \nu = 2s$ and $\dim \tilde{M} = m$. Then we know $2p + q = n$ and $2(p + q + s) = m$.

Remark 2.4. *We know that the dimensions of the distributions $\mathcal{D}$ and ν are real even.*

Now, we recall an adapted frame on $\tilde{M}$. We take a following local orthonormal frame on $\tilde{M}$,

(i) $\{e_1, e_2, ..., e_p, e_{1^*}, e_{2^*}, ..., e_{p^*}\}$ is a local orthonormal frame of $\mathcal{D}$,

(ii) $\{e_{2p+1}, e_{2p+2}, ..., e_{2p+q}\}$ is a local orthonormal frame of $\mathcal{D}^{\perp}$,

(iii) $\{e_{n+q+1}, e_{n+q+2}, ...e_{n+q+s}, e_{(n+q+1)^*}, e_{(n+q+2)^*}, ..., e_{(n+q+s)^*}\}$ is a local orthonormal frame of ν. Then we know

(iv) $\{e_1, ..., e_p, e_{1^*}, ..., e_{p^*}, e_{2p+1}, ..., e_{2p+q}\}$ is a local orthonormal frame of TM,

(v) $\{e_{(2p+1)^*}, ..., e_{(2p+q)^*}, e_{n+q+1}, ..., e_{n+q+s}, e_{(n+q+1)^*}, ..., e_{(n+q+s)^*}\}$ is a local orthonormal frame of $T^{\perp}M$, where $e_{i^*} = Je_i$ for any $i \in \{1, 2, ..., p\}$, $e_{(2p+b)^*} = Je_{2p+a}$ for any $a \in \{1, 2, ..., q\}$ and $e_{(n+q+\alpha)^*} = Je_{n+q+\alpha}$ for any $\alpha \in \{1, 2, ..., s\}$. We call such a local orthonormal frame an *adapted frame* of $\tilde{M}$ ([9]).

3. The Codazzi equation.

In this section, we consider the Codazzi equation in a CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$.

Let M be a CR-submanifold in an l.c.K.-space form $\tilde{M}(c)$. Then the curvature tensor $\tilde{R}$ is given by (1.6). Thus, with respect to an adapted frame, $\{\tilde{R}(X,Y)Z\}^{\perp}$ is written by

$$\begin{cases} 4\tilde{R}_{kjia^*} = 3(P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki}) - P_{ka}\delta_{j^*i} + P_{ja}\delta_{k^*i} + 2P_{ia}\delta_{k^*j}, \\ 4\tilde{R}_{kjir} = 3(P_{kr}\delta_{ji} - P_{jr}\delta_{ki}) - P_{k^*r}\delta_{j^*i} + P_{j^*r}\delta_{k^*i} + 2P_{i^*r}\delta_{k^*j}, \\ 2\tilde{R}_{kjba^*} = -c\delta_{k^*j}\delta_{ba} + P_{k^*j}\delta_{ba} + P_{ba}\delta_{k^*j}, \\ 2\tilde{R}_{kjbr} = P_{b^*r}\delta_{k^*j}, \\ 4\tilde{R}_{kbia^*} = -c\delta_{k^*i}\delta_{ba} - 3P_{ba^*}\delta_{ki} + P_{k^*i}\delta_{ba} + P_{ba}\delta_{k^*i}, \\ 4\tilde{R}_{kbir} = -3P_{br}\delta_{ki} + P_{b^*r}\delta_{k^*i}, \\ 4\tilde{R}_{kcba^*} = 3P_{ka^*}\delta_{cb} + P_{k^*b}\delta_{ca} + 2P_{k^*c}\delta_{ba}, \\ 4\tilde{R}_{kcbr} = 3P_{kr}\delta_{cb}, \\ 4\tilde{R}_{dcba^*} = 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) + P_{d^*b}\delta_{ca} - P_{c^*b}\delta_{da} + 2P_{d^*c}\delta_{ba}, \\ 4\tilde{R}_{dcbr} = 3(P_{dr}\delta_{cb} - P_{cr}\delta_{db}), \end{cases} \tag{3.1}$$

for any $i, j, ..., k \in \{1, 2, ..., 2p\}$, $a, b, ..., d \in \{2p+1, 2p+2, ..., 2p+q = n\}$ and $s, r \in \{n+q+1, n+q+2, m\}$, where we put $\tilde{R}_{\omega\nu\mu\lambda} = \tilde{R}(e_\omega, e_\nu, e_\mu, e_\lambda)$, $P_{\mu\lambda} = P(e_\mu, e_\lambda)$, etc. for any $\omega, \nu, \mu, \lambda \in \{1, 2, ..., n\}$ and we used the properties of P and $\tilde{P}$.

By virtue of (2.4) and (3.1), we obtain

$$\begin{cases} 4\{\tilde{g}((\nabla_k\sigma)_{ji}, e_{a^*}) - \tilde{g}((\nabla_j\sigma)_{ki}, e_{a^*})\} = 3(P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki}) \\ \qquad -P_{ka}\delta_{j^*i} + P_{ja}\delta_{k^*i} + 2P_{ia}\delta_{k^*j}, \\ 4\{\tilde{g}((\nabla_k\sigma)_{ji}, e_r) - \tilde{g}((\nabla_j\sigma)_{ki}, e_r)\} = 3(P_{kr}\delta_{ji} - P_{jr}\delta_{ki}) \\ \qquad -P_{k^*r}\delta_{j^*i} + P_{j^*r}\delta_{k^*i} + 2P_{i^*r}\delta_{k^*j}, \\ 2\{\tilde{g}((\nabla_k\sigma)_{jb}, e_{a^*}) - \tilde{g}((\nabla_j\sigma)_{kb}, e_{a^*})\} = -c\delta_{k^*j}\delta_{ba} \\ \qquad +(P_{k^*j}\delta_{ba} + P_{ba}\delta_{k^*j}), \\ 2\{\tilde{g}((\nabla_k\sigma)_{jb}, e_r) - \tilde{g}((\nabla_j\sigma)_{kb}, e_r)\} = P_{b^*r}\delta_{k^*j}, \\ 4\{\tilde{g}((\nabla_k\sigma)_{bi}, e_{a^*}) - \tilde{g}((\nabla_b\sigma)_{ki}, e_{a^*})\} = -c\delta_{k^*i}\delta_{ba} - 3P_{ba^*}\delta_{ki}, \\ 4\{\tilde{g}((\nabla_k\sigma)_{bi}, e_r) - \tilde{g}((\nabla_b\sigma)_{ki}, e_r)\} = -3P_{br} + P_{k^*r}\delta_{k^*i}, \\ 4\{\tilde{g}((\nabla_k\sigma)_{cb}, e_{a^*}) - \tilde{g}((\nabla_c\sigma)_{kb}, e_r)\} = 3P_{ka^*}\delta_{cb} + P_{kb^*}\delta_{ca} + 2P_{k^*c}\delta_{ba}, \\ 4\{\tilde{g}((\nabla_k\sigma)_{cb}, e_r) - \tilde{g}((\nabla_c\sigma)_{kb}, e_r)\} = 3P_{kr}\delta_{cb}, \\ 4\{\tilde{g}((\nabla_d\sigma)_{cb}, e_{a^*}) - \tilde{g}((\nabla_c\sigma)_{db}, e_{a^*})\} = 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) \\ \qquad +\tilde{P}_{db}\delta_{ca} - P_{c^*b}\delta_{da} + 2P_{dc}\delta_{ba}, \\ 4\{\tilde{g}((\nabla_d\sigma)_{cb}, e_r) - \tilde{g}((\nabla_c\sigma)_{db}, e_r)\} = 3(P_{dr}\delta_{cb} - P_{cr}\delta_{db}), \end{cases} \tag{3.2}$$

for any $i, j, ..., k \in \{1, 2, ..., 2p\}$, $a, b, ..., d \in \{2p+1, 2p+2, ..., 2p+q\}$ and $s, r \in \{n+q+1, n+q+2, m\}$, where we put $\sigma_{\mu\lambda} = \sigma(e_\mu, e_\lambda)$ and $(\nabla_\nu\sigma)_{\mu\lambda} = (\nabla_{e_\nu}\sigma)(e_\mu, e_\lambda)$ for any $\nu, \mu, \lambda \in \{1, 2, ..., n\}$.

Now, we assume that the submanifold M has the symmetric $\nabla\sigma$, that is, σ is a Codazzi type. Then we have from (3.2)

$$\begin{cases} 3(P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki}) - P_{ka}\delta_{j^*i} + P_{ja}\delta_{k^*i} + 2P_{ia}\delta_{k^*j} = 0, \\ 3(P_{kr}\delta_{ji} - P_{jr}\delta_{ki}) - P_{k^*r}\delta_{j^*i} + P_{j^*r}\delta_{k^*i} + 2P_{i^*r}\delta_{k^*j} = 0, \\ c\delta_{k^*j}\delta_{ba} - (P_{k^*j}\delta_{ba} + P_{ba}\delta_{k^*j}) = 0, \\ P_{b^*r}\delta_{k^*j} = 0, \\ c\delta_{k^*i}\delta_{ba} + 3P_{ba^*}\delta_{ki} - P_{k*i}\delta_{ba} - P_{b^*a^*} = 0, \\ P_{br}\delta_{ki} - P_{b^*r}\delta_{k^*i} = 0, \\ 3P_{ka^*}\delta_{cb} + P_{kb^*}\delta_{ca} + 2P_{k^*c}\delta_{ba} = 0, \\ 3P_{kr}\delta_{cb} = 0, \\ 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) + P_{d^*b}\delta_{ca} - P_{c^*b}\delta_{da} + 2P_{d^*c}\delta_{ba} = 0, \\ P_{dr}\delta_{cb} - P_{cr}\delta_{db} = 0. \end{cases} \tag{3.3}$$

By virtue of $(3.3)_{3)}$, we can easily see

$$P_{j^*i^*} = F\delta_{ji}, \quad P_{ba} = G\delta_{ba} \tag{3.4}$$

for any $i, j, ..., k \in \{1, 2, ..., p\}$, $a, b, ..., d \in \{2p+1, 2p+2, ..., 2p+q\}$, where F and G denote the eigenfunctions of P which are given by

$$F = \frac{cq - P_b{}^b}{q}, \quad G = \frac{cp - P_k{}^k}{p}.$$

In particular, for any $i, j, ..., k \in \{1, 2, ..., p\}$, $a, b, ..., d \in \{2p+1, 2p+2, ..., 2p+q\}$ and $s, r \in \{n+q+1, n+q+2, m\}$, the equation (3.3) is written as

$$\begin{cases} P_{ka^*}\delta_{ji} - P_{ja^*}\delta_{ki} = 0, \\ P_{kr}\delta_{ji} - P_{jr}\delta_{ki} = 0, \\ P_{k^*j} = 0, \quad P_{br} = 0, \quad P_{kr} = 0, \\ 3P_{ba^*}\delta_{ki} - P_{k*i}\delta_{ba} = 0, \\ 3P_{ka^*}\delta_{cb} + P_{kb^*}\delta_{ca} + 2P_{k^*c}\delta_{ba} = 0, \\ 3(P_{da^*}\delta_{cb} - P_{ca^*}\delta_{db}) + P_{d^*b}\delta_{ca} - P_{c^*b}\delta_{da} + 2P_{d^*c}\delta_{ba} = 0, \\ P_{dr}\delta_{cb} - P_{cr}\delta_{db} = 0, \end{cases} \tag{3.3'}$$

Using (1.8), the tensor field P satisfies

$$P_{j^*i^*} = P_{ji}, \quad P_{j^*a} = P_{ja^*}, \quad P_{j^*r} = P_{jr^*}, \quad P_{b^*a^*} = P_{ba} \tag{3.5}$$

for any $j, i \in \{1, 2, ..., p\}$, $b, a \in \{2p+1, 2p+2, ..., 2p+q = n\}$ and $r \in \{n+q+1, n+q+2, ..., m\}$.

By virtue of (3.3)′ and the above relations, we obtain

$$P_{j^*i} = 0, \quad P_{ja} = 0, \quad P_{k^*a} = 0, \quad P_{kr} = 0, \quad P_{ba^*} = 0, \quad P_{br} = 0. \tag{3.6}$$

As a result, the tensor field $P_{\mu\lambda}$ is expressed as

$$(P_{\mu\lambda}) = \begin{pmatrix} P_{ji} & P_{ji^*} & P_{ja} & P_{ja^*} & P_{jr} \\ P_{j^*i} & P_{j^*i^*} & P_{j^*a} & P_{j^*a^*} & P_{j^*r} \\ P_{bi} & P_{bi^*} & P_{ba} & P_{ba^*} & P_{br} \\ P_{b^*i} & P_{b^*i^*} & P_{b^*a} & P_{b^*a^*} & P_{b^*r} \\ P_{ri} & P_{ri^*} & P_{ra} & P_{ra^*} & P_{sr} \end{pmatrix} = \begin{pmatrix} P_{ji} & P_{ji^*} & P_{ja} & P_{ja^*} & P_{jr} \\ P_{j^*i} & P_{ji} & P_{j^*a} & P_{j^*a^*} & P_{j^*r} \\ P_{bi} & P_{bi^*} & P_{ba} & P_{ba^*} & P_{br} \\ P_{b^*i} & P_{b^*i^*} & P_{b^*a} & P_{ba} & P_{b^*r} \\ P_{ri} & P_{ri^*} & P_{ra} & P_{ra^*} & P_{sr} \end{pmatrix} \tag{3.7}$$

$$
= \left(\begin{array}{cccc|cccc|ccc}
F & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\
0 & F & 0 & \dots & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & F & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\
\hline
0 & \dots & \dots & 0 & G & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & \dots & 0 & 0 & G & 0 & \dots & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \dots & 0 & 0 & \dots & 0 & G & 0 & \dots & 0 \\
\hline
0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & & P_{sr} &
\end{array}\right).
$$

Thus we have from (3.7)

Theorem 3.1. *In a CR-submanifold M with the symmetric $\nabla\sigma$ in an l.c.K.-space form $\tilde{M}(c)$, the tensor field $P_{\mu\lambda}$ is expressed by* (3.7). *In particular, if M is anti-holomorphic, then the matrix $(P_{\mu\lambda})$ is a diagonal one with two eigenfunctions F and G.*

By virtue of (1.7) and (3.7), we know

$$P_{ji} = -\tilde{\nabla}_j\alpha_i - \alpha_j\alpha_i + \frac{1}{2}\|\alpha\|^2\delta_{ji} = F\delta_{ji}, \tag{3.8}$$

that is,

$$\tilde{\nabla}_j\alpha_i = -\alpha_j\alpha_i + (\frac{1}{2}\|\alpha\|^2 - F)\delta_{ji}. \tag*{(3.8)′}$$

The covariant differentiation of (3.8), (3.8)′ and the Bianchi identity give us

$$\tilde{R}_{kji}{}^A\alpha_A = (\frac{1}{2}\|\alpha\|^2 - F)(\alpha_j\delta_{ki} - \alpha_k\delta_{ji}) - (\frac{1}{2}\tilde{\nabla}_k\|\alpha\|^2 - F_k)\delta_{ji} + (\frac{1}{2}\tilde{\nabla}_j\|\alpha\|^2 - F_j)\delta_{ki}, \tag{3.9}$$

where we put $F_j = \tilde{\nabla}_jF$ and the suffix A run over the range $1, 2, ..., m$.

Next, using (1.6) and (3.7), we find

$$\begin{cases} 4\tilde{R}_{kjih} = (c + 6F)(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) + (c - 2F)\{\tilde{g}(Je_k, e_h)\tilde{g}(Je_j, e_i) \\ \qquad -\tilde{g}(Je_k, e_i)\tilde{g}(Je_j, e_h) - 2\tilde{g}(Je_k, e_j)\tilde{g}(Je_i, e_h)\}, \\ \tilde{R}_{kjia} = 0, \quad \tilde{R}_{kjia^*} = 0, \quad \tilde{R}_{kjir} = 0, \end{cases} \tag{3.10}$$

for any $k, j, i, h \in \{1, 2, ..., 2p\}$, $a \in \{2p + 1, 2p + 2, ..., 2p + q\}$ and $r \in \{n + q + 1, n + q + 2, ..., m\}$.

From $(3.10)_{1)}$, for any $k, j, i, h \in \{1, 2, ..., p\}$, we know

$$4\tilde{R}_{kjih} = \tilde{R}_{k^*j^*i^*h^*} = (c + 6F)(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}). \tag{3.11}$$

On the other hand, we have from (3.10)

$$\begin{aligned} \tilde{R}_{kjiA}\alpha^A = \tilde{R}_{kjih}\alpha^h = \frac{c + 6F}{4}(\alpha_k\delta_{ji} - \alpha_j\delta_{kh}) + \frac{c - 2F}{4}\{\tilde{g}(Je_j, e_i)\tilde{g}(Je_k, e_h)\alpha^h \\ - \tilde{g}(Je_k, e_i)\tilde{g}(Je_j, e_h)\alpha^h - 2\tilde{g}(Je_k, e_j)\tilde{g}(Je_i, e_h)\alpha^h\} \end{aligned} \tag{3.12}$$

for any $A \in \{1, 2, ..., m\}$ and $k, j, i, h \in \{1, 2, ..., 2p\}$.

By virtue of (3.9) and (3.12), we obtain

$$\begin{aligned} (\frac{1}{2}\|\alpha\|^2 + \frac{c + 2F}{4})(\alpha_j\delta_{ki} - \alpha_k\delta_{ji}) + (\frac{1}{2}\tilde{\nabla}_j\|\alpha\|^2 - F_j)\delta_{ki} - (\frac{1}{2}\tilde{\nabla}_k\|\alpha\|^2 - F_k)\delta_{ji} \\ = \frac{2F - c}{4}\{\tilde{g}(Je_j, e_i)\tilde{g}(Je_k, e_h)\alpha^h - \tilde{g}(Je_k, e_i)\tilde{g}(Je_j, e_h)\alpha^h - 2\tilde{g}(Je_k, e_j)\tilde{g}(Je_i, e_h)\alpha^h \end{aligned} \tag{3.13}$$

for any $k, j, i \in \{1, 2, ..., 2p\}$.

In particular, for $k, j, i \in \{1, 2, ..., p\}$ or $k, j, i \in \{p+1, p+2, ..., 2p\}$, the above equation implies

$$(\frac{1}{2}||\alpha||^2 + \frac{c+2F}{4})(\alpha_j\delta_{ki} - \alpha_k\delta_{ji}) + (\frac{1}{2}\tilde{\nabla}_j||\alpha||^2 - F_j)\delta_{ki} - (\frac{1}{2}\tilde{\nabla}_k||\alpha||^2 - F_k)\delta_{ji} = 0. \tag{3.14}$$

Thus, we have from the above equation

$$\frac{1}{2}\tilde{\nabla}_j||\alpha||^2 - F_j = -(\frac{1}{2}||\alpha||^2 + \frac{c+2F}{4})\alpha_j \tag{3.15}$$

for any $j \in \{1, 2, ..., p\}$ if $p \neq 1$. For $k, j, i \in \{p+1, p+2, ..., 2p\}$, we have the same equation with (3.15). Thus, we have (3.15) for any $j \in \{1, 2, ..., 2p\}$, if $p \neq 1$, Thus, by virtue of (3.13) and (3.15), we have

$$(2F - c)\{\tilde{g}(Je_j, e_i)\tilde{g}(Je_k, e_h)\alpha^h - \tilde{g}(Je_k, e_i)\tilde{g}(Je_j, e_h)\alpha^h - 2\tilde{g}(Je_k, e_j)\tilde{g}(Je_i, e_h)\alpha^h\} = 0.$$

From this, we know $F = \frac{c}{2}$ or $\alpha_i = 0$ for any $i \in \{1, 2, ..., 2p\}$. In the case of $\alpha_i = 0$ for any $i \in \{1, 2, ..., 2p\}$, we have from $P_{bi} = 0$ $\tilde{\nabla}_i\alpha_b = 0$, that is, the vector field α_b is parallel in $\mathcal{D}$. Thus we have from the definition of F

Theorem 3.2. *If a CR-submanifold M in an l.c.K.-space form tildeM(c) has the symmetric $\nabla\sigma$ and $p \neq 1$, then we have*

(*i*) *the eigenfunction F of P is constant* ($= \frac{c}{2}$) *or*

(*ii*) *the Lee vector field $\alpha^\sharp$ is orthogonal with to $\mathcal{D}$ and the Lee vector field α_b is parallel in $\mathcal{D}$ for any* $b \in \{2p+1, 2p+2, ..., 2p+q\}$.

Next, we assume that the Lee vector field $\alpha^\sharp$ is orthogonal to $\mathcal{D}$.

From (3.7), we have $P_{ba} = G\delta_{ba}$, that is,

$$\tilde{\nabla}_c\alpha_b = -\alpha_c\alpha_b + (\frac{1}{2}||\alpha||^2 - G)\delta_{cb}. \tag{3.16}$$

Similarly with the last case, we have from (3.16) and Bianchi identity

$$\tilde{R}_{dcb}{}^A\alpha_A = (\frac{1}{2}||\alpha||^2 - G)(\alpha_c\delta_{db} - \alpha_d\delta_{ca}) - (\frac{1}{2}\tilde{\nabla}_d||\alpha||^2 - G_d)\delta_{cb} + (\frac{1}{2}\tilde{\nabla}_c||\alpha||^2 - G_c)\delta_{db}, \tag{3.17}$$

where we put $G_a = \tilde{\nabla}_a G$ for any $a \in \{2p+1, 2p+2, ..., 2p+q = n\}$.

By virtue of (1.6) and (3.7), we have

$$\begin{cases} \tilde{R}_{dcba} = \frac{6G+c}{4}(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}), \\ \tilde{R}_{dcbh} = \tilde{R}_{dcba^*} = \tilde{R}_{dcbr} = 0. \end{cases} \tag{3.18}$$

From (3.18), we have

$$\tilde{R}_{dcba}\alpha^a = \frac{6G+c}{4}(\delta_{cb}\alpha_d - \delta_{db}\alpha_c). \tag{3.19}$$

Thus we have from (3.17) and (3.19)

$$\frac{1}{2}(G + ||\alpha||^2 + \frac{1}{2}c)(\delta_{db}\alpha_c - \delta_{cb}\alpha_d) = (\frac{1}{2}\tilde{\nabla}_d||\alpha||^2 - G_d)\delta_{cb} - (\frac{1}{2}\tilde{\nabla}_c||\alpha||^2 - G_c)\delta_{db}. \tag{3.20}$$

The contraction of the above equation by c and b gives us

$$\frac{1}{2}\tilde{\nabla}_d||\alpha||^2 - G_d = -\frac{1}{2}(G + ||\alpha||^2 + \frac{1}{2}c)\alpha_d, \tag{3.21}$$

if $q \neq 1$.

On the other hand, we know from (3.16)

$$\tilde{\nabla}_d \|\alpha_{\mathcal{D}^\perp}\|^2 = 2(\frac{1}{2}\|\alpha\|^2 - G - \|\alpha_{\mathcal{D}^\perp}\|^2)\alpha_d,$$

where $\alpha_{\mathcal{D}^\perp}$ denotes the $\mathcal{D}^\perp$-component of α. Moreover, we have from (3.7), using $P_{ba^*} = 0$

$$\tilde{\nabla}_d \|\alpha_{J\mathcal{D}^\perp}\|^2 = -2\|\alpha_{J\mathcal{D}^\perp}\|^2\alpha_d,$$

where $\alpha_{J\mathcal{D}^\perp}$ is the $J\mathcal{D}^\perp$-component of α. From the above 2 equations, we obtain

$$\tilde{\nabla}_d \|\alpha_{\mathcal{D}^\perp + J\mathcal{D}^\perp}\|^2 = 2(\frac{1}{2}\|\alpha\|^2 - G - \|\alpha_{\mathcal{D}^\perp + J\mathcal{D}^\perp}\|^2)\alpha_d. \tag{3.22}$$

Now, we assume that the submanifold M is anti-holomorphic ($\nu = \{0\}$), then $\alpha_{\mathcal{D}^\perp + J\mathcal{D}^\perp} = \alpha$. In this case, the equation (3.22) is written as

$$\tilde{\nabla}_d \|\alpha\|^2 = -2(G + \frac{\|\alpha\|^2}{2})\alpha_d.$$

Substituting the above equation into (3.21), we get

$$\tilde{\nabla}_d G = \frac{1}{2}(c - G)\alpha_d \tag{3.23}$$

By the similar calculation with the last case, we obtain

$$\tilde{\nabla}_{d^*} G = \frac{1}{2}(c - G)\alpha_{d^*}. \tag{3.24}$$

Theorem 3.3. *In an anti-holomorphic CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$, if the second fundamental form σ is the Codazzi type, the dimension of $\mathcal{D}^\perp$ is not one and the Lee vector field $\alpha^\sharp$ is orthogonal to $\mathcal{D}$, then the eigen function G satisfies* (3.23) *and* (3.24)*. In particular, if the function G is constant, then $G = c$.*

References

[1] A. Bejancu, *CR*-submanifolds of a Kaehler manifold I, II, *Proc. Amer. Math. Soc.,* **69** (1978), 134 –142 and *Trans. Amer. Math. Soc.,* **250** (1979), 333–345.

[2] A. Bejancu, Geomerty of *CR*-submanifolds, *D. Reidel Publishing Company,* (1986).

[3] V. Bonanzinga and K. Matsumoto, Warped product *CR*-submanifolds in locally conformal Kaehler manifolds, *Periodica Math. Hungarica* **48** (2-2) (2004), 207–221.

[4] V. Bonanzinga and K. Matsumoto, On doubly warped product *CR*-submanifolds in a locally conformal Kaehler manifold,*Tensor (N.S.)* **69** (2008), 76–82.

[5] B. Y. Chen, *CR*-submanifolds of a Kaehler manifold I and II, *J. of Differential Geometry,* **16**, 305–322 and 493–509(1981).

[6] B. Y. Chen, Geometry of submanifolds, *New York, Marcel Dekker,* (1973).

[7] S. Dragomir and L. Ornea, Locally Conformal Kḧler Geometry, *Birkhäuser*, (1998).

[8] T. Kashiwada, Some properties of locally conformal Kähler manifolds, *Hokkaido Math, J.*, **8** (1979), 191–198.

[9] K. Matsumoto, Locally conformal Kähler manifolds and their submanifolds, *Mem. Sec, Acad. Română. Ser. IV* **14** (1991), 7-49(1993).

[10] K. Matsumoto, On *CR*-submanifolds of locally conformal Kähler manifolds I, II, *J. Korean Math.* **21** (1984), 49–61 and *Tensor N, S.,* **45** (1987), 144–150.

[11] K. Matsumoto and Z. Şentürk, Certain twisted product *CR*-submanifolds in a Kaehler manifold, *Proc. of the conference RIGA 2011, Riemannian Geometry and Applications*, 2011.

[12] A. Mihai, Modern Topics in Submanifold theory, *University of Bucharest,* (2006).

[13] I. Vaisman, Locally conformal almost Kähler manifolds, *Israel J, Math.*, **24**(1976), 338–351.

[14] K. Yano, Differential geometry on complex and almost complex spaces, *Pergamon Press,* (1965).